

# Quantum Information Theory

Final exam  
Spring term 2025

From: July 4th, 2023, 9h15  
To: July 4th, 2023, 12h15

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## PHYS-550 – Exam 2024 – SOLUTIONS

- You must answer ALL questions in the short answer section.
- You must answer precisely 2 (out of 3) of the questions in the long answer section.  
Please mark clearly which two you have answered below and **start a new sheet for each of the long answer questions.**
- **Write your solutions in the indicated space.** Scrap paper will not be corrected.
- A simple calculator (without internet access) is allowed.
- Please write your name on the top right corner of each sheet you use.
- Good luck! Enjoy!

NAME STICKER GOES HERE

Short answers: Problem 1	/ 50
Problem A: YES or NO	/ 25
Problem B: YES or NO	/ 25
Problem C: YES or NO	/ 25
<b>Total</b>	/100

# Shorter (easier) questions

## 1. Partial Trace and Fidelity

- Explain why the partial trace operation is important in quantum theory. (3 marks).
- Show that partial trace is a quantum channel by writing it in Kraus form. (3 marks).
- Define the fidelity between two states. What is its operational meaning? (3 marks).
- Compute the fidelity between the states

$$\begin{aligned} |\psi\rangle_{ABC} &= \frac{1}{\sqrt{3}}(|000\rangle + |110\rangle + |220\rangle) \\ |\phi\rangle_{ABC} &= \frac{1}{\sqrt{6}}(2|000\rangle + |010\rangle + |201\rangle) \end{aligned} \tag{1}$$

(1 mark).

- Compute the reduced states  $\rho_A^\psi$  and  $\rho_A^\phi$ , of  $|\psi\rangle_{ABC}$  and  $|\phi\rangle_{ABC}$  respectively on system A. (2 marks)
- What is the fidelity between  $\rho_A^\psi$  and  $\rho_A^\phi$ ? (3 marks)

## 2. Measurement.

Spock hands you a quantum state and promises that it is either  $|\phi_1\rangle = |0\rangle$  or  $|\phi_2\rangle = |-\rangle$ .

- State a POVM-measurement that perfectly distinguishes the states *some* of the time, is inconclusive at others, but never makes a mistake.

(3 marks).

- How would you find the *optimal* POVM elements that perfectly distinguishes the states *some* of the time, is inconclusive at others, but never makes a mistake? (You don't have to find this POVM but explain how you would find it/prove that it is optimal.)

(3 marks)

## 3. 'Measure-and-update' channel.

Consider a measure-and-update channel  $\mathcal{E}(\rho) = \sum_k |k\rangle\langle k| \otimes B_k \rho B_k^\dagger$  where  $\rho$  is the initial state to be measured,  $\{|k\rangle\}$  is an orthonormal basis and  $\{B_k\}$  are a set of Kraus operators.

- Write down the Kraus operators for the channel  $\mathcal{E}$ . Explain why this is called the 'Measure-and-update' channel. (3 marks)

Now consider the following choice in operators  $\{B_k\}$ :

$$B_1 = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} \sqrt{1-\lambda} & 0 \\ 0 & 1 \end{pmatrix}, \tag{2}$$

where  $\lambda$  is real and  $0 \leq \lambda \leq 1$ .

Further suppose the initial state is the  $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ .

b) What is the probability of measuring outcomes 1 and 2? What are the corresponding output states of the system that is measured? (5 marks)

Suppose you now instead have the operators  $\{B'_k\}$ :

$$B'_1 = \begin{pmatrix} 0 & 0 \\ \sqrt{\lambda} & 0 \end{pmatrix}, \quad B'_2 = \begin{pmatrix} \sqrt{1-\lambda} & 0 \\ 0 & 1 \end{pmatrix}, \quad (3)$$

where  $\lambda$  is defined above.

c) What are the probabilities and output states in this case? (4 marks)

d) What do you conclude from comparing your answers in (b) and (c)? (1 mark)

#### 4. Entropy.

a) Show that  $S(U\rho U^\dagger) = S(\rho)$  for any unitary  $U$ . (1 mark).

b) For a composite system  $AB$  in a pure state  $|\psi\rangle_{AB}$  show that  $S(\rho_A) = S(\rho_B)$ .  
(3 marks).

c) Prove that  $S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B)$  (2 marks).

#### 5. LOCC transformations.

What deterministic LOCC transformations (one way, two way or no-way) are possible between the states:

a)  $|\psi_-\rangle$  and  $\alpha|00\rangle + \beta|11\rangle$ .

(2 marks)

b)  $|\phi_1\rangle = \sqrt{\frac{1}{14}}(|00\rangle + 3|11\rangle + 2|22\rangle)$  and  $|\phi_2\rangle = \sqrt{\frac{1}{11}}(|00\rangle + |11\rangle + 3|22\rangle)$ .

(6 marks)

c) With reference to the resource theory of entanglement comment on what you can conclude from your answers to a) and b).

(2 marks)

# Longer (harder) questions

Please **pick 2 questions** to attempt - mark your choices clearly on the cover sheet.

Start a new sheet for each question.

## Question A - Shot Noise

This question will compare the effect of shot noise for three approaches to estimating  $\text{Tr}[\rho H]$ , i.e., the energy of a state  $\rho$  with Hamiltonian  $H = \sum_{i=1}^M \alpha_i P_i$  where the  $P_i$  are Pauli operators.

1. First suppose you measure each Pauli term individually using  $N/M$  shots for each Pauli term. Write down an estimator for this measurement strategy. Show that the estimator is unbiased and compute its variance. Find an upper bound for the variance of the estimator in terms of a norm of the vector  $\vec{\alpha}$ .

(5 marks)

Estimator:  $\hat{X} = \frac{1}{\tilde{N}} \sum_{j=1}^{\tilde{N}} \sum_{i=1}^M \alpha_i \lambda_i^{(j)}$  where  $\tilde{N} = N/M$ .

(1 mark)

To show that this is unbiased we compute

$$\langle \hat{X} \rangle = \frac{1}{\tilde{N}} \sum_{j=1}^{\tilde{N}} \sum_{i=1}^M \alpha_i \langle \lambda_i^{(j)} \rangle = \frac{1}{\tilde{N}} \sum_{j=1}^{\tilde{N}} \sum_{i=1}^M \alpha_i \langle P_i \rangle = \frac{1}{\tilde{N}} \sum_{j=1}^{\tilde{N}} \langle H \rangle = \langle H \rangle.$$

(1 mark)

To compute the variance we use

$$\text{Var}(\hat{X}) = \text{Var} \left( \sum_i \alpha_i \hat{X}_i \right) = \sum_i |\alpha_i|^2 \text{Var}(\hat{X}_i)$$

where we have defined the estimator  $\hat{X}_i = \frac{1}{\tilde{N}} \sum_{j=1}^{\tilde{N}} \lambda_i^{(j)}$  and used the fact that each of these estimators is itself a random variable and independent from the others. The rest of the calculation is now straightforward:

$$\begin{aligned} \text{Var}(\hat{X}) &= \sum_i |\alpha_i|^2 \text{Var}(\hat{X}_i) = \sum_i |\alpha_i|^2 \frac{\text{Var}(P_i)}{\tilde{N}} \\ &= \sum_i |\alpha_i|^2 \frac{\langle P_i^2 \rangle - \langle P_i \rangle^2}{\tilde{N}} = \sum_i |\alpha_i|^2 \frac{1 - \langle P_i \rangle^2}{\tilde{N}} \leq \sum_i |\alpha_i|^2 \frac{1}{\tilde{N}} = \frac{M \|\vec{\alpha}\|_2^2}{N} \end{aligned}$$

(3 marks)

2. Next suppose you instead use  $N_i = p_i N$  shots for each Pauli term where  $p_i = |\alpha_i| / \sum_i |\alpha_i|$ . Why intuitively might this be a better strategy than the previous one?

Because we use more shots to measure the higher weight terms which contribute more to the term we are trying to compute. (1 mark)

Write down an estimator for this measurement strategy. Show that the estimator is unbiased and compute its variance. Find an upper bound for the variance of the estimator in terms of a norm of the vector  $\vec{\alpha}$ .

(5 marks)

Intuition: Because we use more shots to measure the higher weight terms which contribute more to the term we are trying to compute. (1 mark)

Estimator:  $\hat{X} = \sum_{i=1}^M \alpha_i \hat{X}_i$  where  $\hat{X}_i = \frac{1}{N_i} \sum_{j=1}^{N_i} \lambda_i^{(j)}$  with  $N_i = p_i N$ . (1 mark)

Unbiased: (Same as above but  $\tilde{N}$  replaced with  $N$ . (1 mark)

Variance: As the  $\hat{X}_i$  are again independent random variables we can follow the same steps as above to get:

$$\text{Var}(\hat{X}) = \sum_i |\alpha_i|^2 \frac{\text{Var}(P_i)}{N_i} \leq \sum_i |\alpha_i|^2 \frac{1}{N_i} = \sum_i |\alpha_i|^2 \frac{\|\alpha\|_1}{\alpha_i N} = \frac{\|\alpha\|_1^2}{N} \quad (4)$$

(2 marks)

3. Discuss which of these strategies is best for estimating  $\text{Tr}[\rho H]$  if only taking into account shot noise? Construct some example Hamiltonians to illustrate your arguments.

(5 marks)

Although  $\|\vec{\alpha}\|_1 \geq \|\vec{\alpha}\|_2$  the extra factor of  $M$  for method 1 means that the convergence is generally worse for method 1. (1 mark)

As a quick sanity check the methods should agree when the  $\alpha_i$ s are all equal. For example, consider  $H = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_x)$ . The upper bound on the variance for both methods is  $2/N$ .

But generally for uneven terms 2 will have a lower upper bound on the variance than method 1. For example, say  $\alpha_0 = \frac{1}{\sqrt{10}}$  and  $\alpha_1 = \frac{\sqrt{9}}{\sqrt{10}}$  then for method 1 the variance is  $2/N$  but for method 2 the variance is roughly  $1.6/N$ .

(4 marks)

4. Finally, suppose you instead have a source of classical randomness and with probability  $p_i = |\alpha_i| / \sum_i |\alpha_i|$  measure Pauli  $P_i$ .

Again, write down an estimator for this measurement strategy. Show that the estimator is unbiased and compute its variance. And find an upper bound for the estimator in terms of a norm of the vector  $\vec{\alpha}$ .

(5 marks)

Estimator: We measure  $P_i$  with probability  $p_i$  to give  $\lambda_i^{(j)}$ . This leads to  $\hat{X} = \frac{1}{N} \sum_{j=1}^N \hat{x}_j$  where  $\hat{x}_j = \frac{\alpha_i \lambda_i^{(j)}}{p_i}$ . (1 mark)

Unbiased: The average of this estimator is

$$\langle \hat{X} \rangle = \frac{1}{N} \sum_j \langle x_j \rangle = \langle x_j \rangle = \sum_i p_i \frac{\alpha_i \langle \lambda_i^{(j)} \rangle}{p_i} = \sum_i \alpha_i \langle P_i \rangle = \langle H \rangle, \quad (5)$$

where in a slight abuse of notation we use  $\langle \dots \rangle$  to first denote the average over both the sampling of the Pauli to measure and the measurement of the Pauli, and later to just measure the Pauli. (1 mark)

Variance:

$$\text{Var}(\hat{X}) \leq \frac{1}{N} \left\langle \frac{\alpha_i^2 \lambda_i^{(j)2}}{p_i^2} \right\rangle = \frac{1}{N} \sum_i p_i \frac{\alpha_i^2 \langle P_i^2 \rangle}{p_i^2} = \frac{1}{N} \|\vec{\alpha}\|_1 \sum_i \frac{\alpha_i^2}{|\alpha_i|} = \frac{\|\vec{\alpha}\|_1^2}{N} \quad (6)$$

(3 marks)

Discuss how this method compares to the others? When might you choose to use it?

(2 marks)

The scaling (at least of the upper bound) is the same as method 2. You might choose to use it if  $N_i = p_i N$  is not an integer and so will induce rounding errors.

5. Suggest two other methods you might use to estimate  $\text{Tr}[\rho H]$ . What are the advantages / disadvantages of these methods? (3 marks)
- 1) Find a circuit to rotate into the eigenbasis of the Hamiltonian and measure in that basis (not possible in general!).
  - 2) Group together terms that commute and measure those simultaneously.

## Question B - Matrix Norms

Say you have some arbitrary channel  $\mathcal{E}$  and you want to bound  $\|\mathcal{E}(O)\|_2$  where  $O$  is an arbitrary observable. You can assume that  $\mathcal{E}$  maps a  $d_S$  dimensional system to a  $d_S$  dimensional system.

We will assume that this channel can be written in Kraus form as  $\mathcal{E}(\dots) = \sum_{k=1}^M B_k \dots B_k^\dagger$ .

You can also assume that the channel has the Stinespring dilation using a  $d_E$  dimensional ancilla system.

(Don't be intimidated by the large number of marks for 2 and 3 questions. These are short calculations. But they might require spotting a few tricks.)

1. How is  $\|\rho_1 - \rho_2\|_1$  related to the maximum probability of distinguishing  $\rho_1$  and  $\rho_2$  (assuming you are given  $\rho_1$  or  $\rho_2$  with equal probability)? Prove this relationship.

(7 marks)

Using the POVM  $\mathcal{M} = \{M_1, M_2 = \mathbb{I} - M_1\}$  such that  $M_i$  is associated with outcome  $i$  corresponding to the case where the state is  $\rho_i$ . Thus, the probability of guessing the state correctly given that both are given with equal probabilities is given by

$$\begin{aligned} p_{\text{correct guess}} &= \Pr(\text{measure 1} | \text{input } \rho_1) \Pr(\text{input } \rho_1) + \Pr(\text{measure 2} | \text{input } \rho_2) \Pr(\text{input } \rho_2) \\ &= \text{Tr}[M_1 \rho_1] \frac{1}{2} + \text{Tr}[M_2 \rho_2] \frac{1}{2} \\ &= \frac{1}{2} (\text{Tr}[M_1 \rho_1] + \text{Tr}[(\mathbb{I} - M_1) \rho_2]) \\ &= \frac{1}{2} (1 + \text{Tr}[M_1 (\rho_1 - \rho_2)]) , \end{aligned}$$

where we used  $M_2 = \mathbb{I} - M_1$  in the third equality and  $\text{Tr}[\rho_2] = 1$  in the last one. From this expression, we see that maximising  $p_{\text{correct guess}}$  is equivalent to maximising  $\text{Tr}[M_1 (\rho_1 - \rho_2)]$  (under the POVM criterion for  $\mathcal{M}$ ). Now, we consider the eigendecomposition  $\rho_1 - \rho_2 = \sum_i \lambda_i |\lambda_i\rangle \langle \lambda_i|$  such that each  $\lambda_i$  is real (from the hermitian properties of quantum states). We split it by considering the positive and negative eigenvalues separately such as  $P_+ = \sum_{i, \lambda_i < 0} \lambda_i |\lambda_i\rangle \langle \lambda_i|$  and  $P_- = \sum_{i, \lambda_i > 0} |\lambda_i| |\lambda_i\rangle \langle \lambda_i|$  which leads to  $\rho_1 - \rho_2 = P_+ - P_-$ . Thus we have  $\text{Tr}[M_1 (\rho_1 - \rho_2)] = \text{Tr}[M_1 P_+] - \text{Tr}[M_1 P_-]$ , and as both  $P_-$ ,  $P_+$  and  $M_1$  are positive operators (and  $\mathcal{M}$  forms a POVM), we can see that this quantity is maximised when  $M_1$  is a projector onto the positive eigenspace of  $\rho_1 - \rho_2$  i.e. such that  $\text{Tr}[M_1 P_-] = 0$  and  $\text{Tr}[M_1 P_+] = \text{Tr}[P_+] = \sum_{i, \lambda_i > 0} \lambda_i$ . This leads to the maximal probability of correct guess given by  $p_{\text{max}} = \frac{1}{2} (1 + \text{Tr}[P_+])$ . On the other side, we have  $0 = \text{Tr}(\rho_1 - \rho_2) = \text{Tr}[P_+] - \text{Tr}[P_-]$  which implies  $\text{Tr}[P_+] = \text{Tr}[P_-]$ . Thus,  $\|\rho_1 - \rho_2\|_1 = \text{Tr}[P_+] + \text{Tr}[P_-] = 2\text{Tr}[P_+]$  which finally leads to  $p_{\text{max}} = \frac{1}{2} (1 + \frac{\|\rho_1 - \rho_2\|_1}{2})$ .

2. Derive the tightest upper bound you can for  $\|\mathcal{E}(O)\|_2$  in terms  $M$ ,  $d_S$ , and the norm of  $O$  (you can pick the norm).

Hint- you may want to derive multiple bounds depending on whether  $O$  is a projector or a (sum of) Pauli operators.

(6 marks)

First, let us derive a straightforward bound by using triangle inequality and sub-multiplicativity (i.e.  $\|AB\| \leq \|A\| \|B\|$ ).

$$\begin{aligned} \|\mathcal{E}(O)\|_2 &\leq \sum_{k=1}^M \|B_k O B_k^\dagger\|_2 \\ &\leq \sum_{k=1}^M \|B_k\|_2 \|O\|_2 \|B_k^\dagger\|_2 \\ &= \|O\|_2 \sum_{k=1}^M \|B_k\|_2^2, \end{aligned}$$

where the last equality is obtained by using  $\|A\| = \|A^\dagger\|$ . Now, we have  $\sum_k \|B_k\|_2^2 = \sum_k \text{Tr}(B_k^\dagger B_k) = \text{Tr}(\mathbb{I}) = d_S$ , thus an upper-bound would be  $\|\mathcal{E}(O)\|_2 \leq d_S \|O\|_2$ . Now, if  $O$  is a Pauli, then  $\|O\|_2 = \sqrt{d_S}$  and if  $O$  is a projector onto a quantum state then  $\|O\|_2 = 1$ . In the case where  $O$  is a Pauli, we rather want to get a bound that depends on the spectral (infinity) norm of  $O$ . However, we have  $\|O\|_2 \leq \sqrt{d_S} \|O\|_\infty$  which becomes an equality in the case where  $O$  is a Pauli.

3. Derive the tightest possible upper bound you can for  $\|\mathcal{E}(O)\|_2$  which is independent of  $M$ . It may, or may not, also depend on  $d_E$  and/or  $d_S$ . It will depend on the norm of  $\|\mathcal{E}(O)\|_2$  (you can pick the norm).

Hint- you may want to derive multiple bounds depending on whether  $O$  is a projector or a (sum of) Pauli operators.

(8 marks)

We already have a bound that is independent on  $M$ . However, we can try to derive another bound from the Stinespring dilation of the channel. Let us assume a Stinespring dilation unitary  $U$  acting on a  $d_S d_E$  dimensional space such that  $\mathcal{E}(O) = \text{Tr}_E[U(O \otimes |0\rangle\langle 0|)U^\dagger]$ . Now, you can rederive the result seen in the class problem using generalised Pauli operators which leads to  $\|\text{Tr}_E(Q)\|_p \leq d_E^{\frac{p-1}{p}} \|Q\|_p$  for any operator  $Q$  acting on the composite systems  $S$  and  $E$ . Replacing  $Q = U(O \otimes |0\rangle\langle 0|)U^\dagger$  in previous

inequality leads to

$$\|\mathcal{E}(O)\|_2 = \|\text{Tr}_E[U(O \otimes |0\rangle\langle 0|)U^\dagger]\|_2 \quad (7)$$

$$\leq \sqrt{d_E} \|U(O \otimes |0\rangle\langle 0|)U^\dagger\|_2 \quad (8)$$

$$= \sqrt{d_E} \|O \otimes |0\rangle\langle 0|\|_2 \quad (9)$$

$$= \sqrt{d_E} \|O\|_2, \quad (10)$$

where the second equality follows from unitary invariance and the last one is trivial to show.

4. Discuss which of these bounds is tighter?

(4 marks)

Regarding the first bound given by  $\|\mathcal{E}(O)\|_2 \leq d_S \|O\|_2$  and the second one given by  $\|\mathcal{E}(O)\|_2 \leq \sqrt{d_E} \|O\|_2$ , we need to compare  $d_E$  and  $d_S^2$ . Any channel can be represented using at least  $d_S^2$  Kraus operators (i.e.  $M \leq d_S^2$ ) and a corresponding Stinespring unitary can be found with an ancilla system of dimension  $d_E = M$ . However, some channels might be expressed with less than  $d_S^2$  Kraus operators and in this case we can find a Stinespring unitary  $U$  such that  $M = d_E < d_S^2$  so the second bound (with  $d_E$ ) should be better than the first one in general (or equivalent in the worst case) i.e.  $\|\mathcal{E}(O)\|_2 \leq \sqrt{d_E} \|O\|_2 \leq d_S \|O\|_2$ .

Bonus (genuine ongoing research question): what is the tightest bound you can obtain on  $\|\mathcal{E}^*(P)\|_2$  where  $\mathcal{E}^*$  is the adjoint of a channel and  $P$  is a Pauli? (Note: the adjoint of a channel is not necessarily a channel.)

## Question C - Superoperators to detect entanglement

1. Define the maximally entangled state  $|\Omega\rangle = |\text{vec}(\mathbb{I})\rangle$ . Show that  $|\Omega\rangle\langle\Omega|^{T_A} = \text{SWAP}$ .  
(2 marks)

$$|\Omega\rangle\langle\Omega|^{T_A} = \sum_{i,j=1}^d |ii\rangle\langle jj|^{T_A} \quad (11)$$

$$= \sum_{i,j=1}^d |i\rangle\langle j|^T \otimes |i\rangle\langle j| \quad (12)$$

$$= \sum_{i,j=1}^d |j\rangle\langle i| \otimes |i\rangle\langle j| \quad (13)$$

$$= \sum_{i,j=1}^d |ji\rangle\langle ij| \quad (14)$$

$$= \text{SWAP} , \quad (15)$$

which is also true if we take partial transpose with respect to subsystem  $B$  i.e.  $|\Omega\rangle\langle\Omega|^{T_B} = \text{SWAP}$ .

2. Show that  $\text{Tr}[A \otimes B \text{SWAP}] = \text{Tr}[AB]$ .

(2 marks)

Using  $A = \sum_{i,j} a_{ij} |i\rangle\langle j|$  and  $B = \sum_{i,j} b_{ij} |i\rangle\langle j|$ , we have

$$\text{Tr}[A \otimes B \text{SWAP}] = \sum_{i,j,i',j'} a_{ij} b_{i'j'} \text{Tr}[|ii'\rangle\langle jj'| \text{SWAP}] \quad (16)$$

$$= \sum_{i,j,i',j'} a_{ij} b_{i'j'} \langle jj'| \text{SWAP} |ii'\rangle \quad (17)$$

$$= \sum_{i,j,i',j'} a_{ij} b_{i'j'} \langle jj'|i'i\rangle \quad (18)$$

$$= \sum_{i,j} a_{ij} b_{ji} \quad (19)$$

$$= \text{Tr}[AB] \quad (20)$$

where we used  $\text{SWAP}|ii'\rangle = |i'i\rangle$  in the third equality and the last equality is quite straightforward as  $AB = \sum_{i,j,k} a_{ij} b_{jk} |i\rangle\langle k|$  then taking the trace leads to the desired result.

3. For any map  $\mathcal{E}$  can implicitly define the notion of a ‘dual map’  $\mathcal{E}^*$  such that

$$\text{Tr}[\mathcal{E}(\rho)B] = \text{Tr}[\rho\mathcal{E}^*(B)] \quad (21)$$

for all states  $\rho$  and Hermitian operators  $B$ .

For a unitary channel what form does  $\mathcal{E}^*$  take?

(1 marks)

A unitary channel is of the form  $\mathcal{E}(X) = UXU^\dagger$  where  $U$  is unitary. Then from the definition of the dual map and using cyclicity of the trace, we have

$$\text{Tr}[\mathcal{E}(\rho)B] = \text{Tr}[U\rho U^\dagger B] \quad (22)$$

$$= \text{Tr}[\rho U^\dagger B U] \quad (23)$$

$$= \text{Tr}[\rho\mathcal{E}^*(B)] , \quad (24)$$

where the last equality is imposed by the above definition for the dual map. So, in this case the dual map is given by  $\mathcal{E}^*(X) = U^\dagger X U$ .

4. Assuming that every positive superoperator  $\mathcal{E}$  has a decomposition  $\mathcal{E}(X) = \sum_i A_i X B_i^\dagger$ . Derive an explicit expression for  $\mathcal{E}^*$  in terms of  $A_i$  and  $B_i$ .

(5 marks)

This proof is similar to the unitary channel case, but here we additionally use the linearity of the trace such that

$$\text{Tr}[\mathcal{E}(\rho)B] = \sum_i \text{Tr}[A_i \rho B_i^\dagger B] \quad (25)$$

$$= \sum_i \text{Tr}[\rho B_i^\dagger B A_i] \quad (26)$$

$$= \text{Tr}[\rho \sum_i B_i^\dagger B A_i] \quad (27)$$

$$= \text{Tr}[\rho\mathcal{E}^*(B)] , \quad (28)$$

which leads to  $\mathcal{E}^*(X) = \sum_i B_i^\dagger X A_i$  (where the last equality is imposed by the definition of the dual map).

5. The Choi state associated with an operator  $\mathcal{E}$  is defined as  $J(\mathcal{E}) = \mathcal{E} \otimes \mathbb{I}(|\Omega\rangle\langle\Omega|)$ .

Prove that

$$\text{Tr}[(B \otimes \sigma^T)J(\mathcal{E})] = \text{Tr}[\mathcal{E}(\sigma)B] \quad (29)$$

(6 marks)

$$\text{Tr}[(B \otimes \sigma^T)J(\mathcal{E})] = \text{Tr}[(B \otimes \sigma^T)\mathcal{E} \otimes \mathbb{I}(|\Omega\rangle\langle\Omega|)] \quad (30)$$

$$= \text{Tr}[(\mathcal{E}^*(B) \otimes \sigma^T)|\Omega\rangle\langle\Omega|] \quad (31)$$

$$= \text{Tr}[(\mathcal{E}^*(B) \otimes \sigma)(|\Omega\rangle\langle\Omega|)^{TB}] \quad (32)$$

$$= \text{Tr}[(\mathcal{E}^*(B) \otimes \sigma)\text{SWAP}] \quad (33)$$

$$= \text{Tr}[\mathcal{E}^*(B)\sigma] \quad (34)$$

$$= \text{Tr}[\mathcal{E}(\sigma)B] , \quad (35)$$

where the second equality follows from the definition of the dual map (it can be shown easily from cyclicity and linearity of the trace), the third equality follows from trace invariance with respect to transposed or partial transposed operator, the fourth equality is proven in 1., the fifth equality is proven in 2., and the last equality follows from the definition of the dual map again.

6. We now define the super-operator  $\Phi_{\mathcal{J}}(\rho) = \text{Tr}_2[(\mathbb{I} \otimes \rho^T)\mathcal{J}]$

Prove that  $\Phi_{\mathcal{J}(\mathcal{E})}(\rho) = \mathcal{E}(\rho)$  and  $\mathcal{J}(\Phi_{\mathcal{J}(\mathcal{E})}) = \mathcal{J}(\mathcal{E})$ .

How does this connect to Eq. (29)?

(6 marks)

Let's prove  $\Phi_{\mathcal{J}(\mathcal{E})}(\rho) = \mathcal{E}(\rho)$  first (which is the hard part of the question). Using the definitions above, we have

$$\Phi_{\mathcal{J}(\mathcal{E})}(\rho) = \text{Tr}_2[(\mathbb{I} \otimes \rho^T)\mathcal{J}(\mathcal{E})] \quad (36)$$

$$= \text{Tr}_2[(\mathbb{I} \otimes \rho^T)\mathcal{E} \otimes \mathbb{I}(|\Omega\rangle\langle\Omega|)] \quad (37)$$

$$= \sum_{i,j} \text{Tr}_2[(\mathbb{I} \otimes \rho^T)\mathcal{E}(|i\rangle\langle j|) \otimes |i\rangle\langle j|] \quad (38)$$

$$= \sum_{i,j} \text{Tr}_2[\mathcal{E}(|i\rangle\langle j|) \otimes (\rho^T|i\rangle\langle j|)] \quad (39)$$

$$= \sum_{i,j} \mathcal{E}(|i\rangle\langle j|)\langle j|\rho^T|i\rangle \quad (40)$$

$$= \sum_{i,j} \mathcal{E}(|i\rangle\langle j|)\langle i|\rho|j\rangle \quad (41)$$

$$= \mathcal{E}\left(\sum_{i,j} \langle i|\rho|j\rangle|i\rangle\langle j|\right) \quad (42)$$

$$= \mathcal{E}(\rho) , \quad (43)$$

the third and seventh equalities follows from the linearity of  $\mathcal{E}$ , the fifth equality is obtained by tracing out the second subsystem explicitly, then we used  $\langle i|\rho|j\rangle = \langle j|\rho^T|i\rangle$  in the sixth equality.

Now, we can easily prove the second equality as follows

$$\mathcal{J}(\Phi_{\mathcal{J}(\mathcal{E})}) = \Phi_{\mathcal{J}(\mathcal{E})} \otimes \mathbb{I}(|\Omega\rangle\langle\Omega|) \quad (44)$$

$$= \sum_{i,j} \Phi_{\mathcal{J}(\mathcal{E})}(|i\rangle\langle j|) \otimes |i\rangle\langle j| \quad (45)$$

$$= \sum_{i,j} \mathcal{E}(|i\rangle\langle j|) \otimes |i\rangle\langle j| \quad (46)$$

$$= \mathcal{E} \otimes \mathbb{I}(|\Omega\rangle\langle\Omega|) \quad (47)$$

$$= \mathcal{J}(\mathcal{E}) , \quad (48)$$

where the third equality was just proven above i.e.  $\Phi_{\mathcal{J}(\mathcal{E})}(\rho) = \mathcal{E}(\rho)$  and the rest is only definitions and linearity of CPTP maps. We can see that setting  $B = \mathbb{I}$  in Eq. (29) is the result obtained after taking the trace of the equality  $\Phi_{\mathcal{J}(\mathcal{E})}(\rho) = \mathcal{E}(\rho)$ .

7. Use your answers above to explain the connection between SWAP and the Peres-Horodecki criterion?

(3 marks)

The SWAP operator is the entanglement witness corresponding to the Peres-Horodecki criterion. That is,  $\text{Tr}[\rho \text{SWAP}] < 0$  implies  $\rho^{T_B}$  has a negative eigenvalue. To see this note that  $\text{Tr}[\rho \text{SWAP}] = \text{Tr}[\rho^{T_B} |\Omega\rangle\langle\Omega|]$  (from 1. and invariance of the trace operation with respect to partial transpose). Writing  $\rho^{T_B}$  in terms of its eigendecomposition it follows that if  $\text{Tr}[\rho \text{SWAP}] < 0$  then  $\rho^{T_B}$  has a negative eigenvalue.